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Quasi-Inertial Stabilization of the AAP 1/2 Cluster Configuration

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ABSTRACT

For certain space missions such as AAP 1/2, a long duration requirement for solar pointing may exist, if fixed solar panel arrays are considered for meeting a substantial portion of the electrical power requirements. In this report an approach for stabilizing the AAP 1/2 cluster configuration in a Quasi-Inertial attitude mode has been investigated. In this mode the spacecraft principal axis of minimum moment of inertia oscillates in the orbital plane with an average orientation normal to the sun line.

The results of this study indicate that the Quasi-Inertial approach offers the possibility of obtaining high efficiency from fixed solar panels with relatively low RCS fuel consumption. The degree of degradation in solar panel efficiency due to attitude motion of the spacecraft about the sun line depends on the parameter α, the ratio of peak aerodynamic torque to peak gravity gradient torque. The degradation in efficiency is only 21% for the largest value of α expected (α = 0.4) and only 3.1% for α = 0.12 which is typical.

The analysis includes an exact solution of the spacecraft equations of motion, an evaluation of typical initialization requirements for the mode and an estimate of the RCS fuel consumption. The results indicate that an RCS fuel requirement of only 300 lbs./mo. appears possible as compared with 5800 lbs./mo. for maintaining an inertial orientation with standard limit cycle control.

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1.0 INTRODUCTION

One of the objectives in the Apollo Applications Program is to achieve the capability of supporting man in space for as long as one year. An approach for meeting this goal is the use of cluster configurations such as the Orbital Assembly* currently being planned by NASA for the AAP 1/2 and 3/4 earth orbital missions. Attitude control of such configurations can lead to appreciable momentum storage and/or RCS fuel requirements, if an inertial orientation is maintained for long periods, since such large spacecraft encounter substantial gravity-gradient and aerodynamic torque disturbances.

In the AAP 3/4 mission, in which solar astronomy experiments will be conducted with the ATM/LM vehicle docked to the Orbital Assembly, control moment gyros (CMG's) are available for precise attitude control. An inertial vehicle orientation has been proposed for this mission,** which takes maximum advantage of the CMG as a momentum exchange device. In this mode the spacecraft principal axis of minimum moment of inertia is oriented in the orbital plane normal to the sun line. The ATM experiment line-of-sight is normal to this axis so that the effect of orbit inclination to the ecliptic plane and nodal regression can be overcome with appropriate roll displacement about the principal axis of minimum moment of inertia. The advantage of this mode is that the components of gravity-gradient torque are periodic except for a small bias component due to vehicle asymmetry.

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*The basic Orbital Assembly is comprised of an S-IVB Workshop, Airlock Module (AM), Multiple Docking Adapter (MDA) and CSM.

**G. M. Anderson and W. W. Hough, "Hard-Docked ATM Experiment Carrier", Bellcomm Technical Memorandum TM-66-1022-01, November 18, 1966.

In other AAP missions a long duration requirement for solar pointing may also exist since the use of fixed solar panel arrays is being considered for meeting at least part of the electrical power requirements. The pointing requirement is not as severe for solar panels however, since a misalignment of a few degrees does not significantly degrade the performance. When CMG's are not available for attitude control as in the AAP 1/2 mission, counteracting the gravity-gradient and aerodynamic disturbance torque experienced in an inertial pointing mode leads to a substantial RCS fuel requirement.

The purpose of this report is to describe a Quasi-Inertial attitude mode which is similar to the ATM mode in that the spacecraft orientation is nominally the same. The difference is that the spacecraft principal axis of minimum moment of inertia oscillates in the orbit plane about an orientation normal to the sun line. Solution of the equations of motion is carried out to determine the maximum deviation from the nominal orientation of the spacecraft as well as the initial conditions required to establish the mode. The effect of aerodynamic torque is also considered and the results are supported by results obtained from a digital computer simulation.

2.0 DESCRIPTION OF THE PROBLEM

In this section the geometry associated with the ATM mode is reviewed and basic features of a Quasi-Inertial stabilization mode are outlined.

2.1 Geometrical Aspects

The general attitude orientation of the spacecraft principal axes (xyz) with respect to an inertial reference coordinate system (XYZ) can be specified by the Euler rotation sequence (Ψ, θ, Φ) shown in Figure 1. The local vertical position of the spacecraft in orbit is defined by the angle η. The spacecraft principal axis of minimum moment of inertia is assumed to be the x axis. In the ATM mission the x axis is maintained in the orbital plane ($\theta = 0$) and positioned normal to the sun line (Ψ rotation). The experiment line-of-sight assumed parallel to the y axis is aligned toward the sun by roll motion about the x axis (Φ rotation). The angles Ψ and Φ are varied periodically to account for nodal regression and season. The required variation in Φ is limited to the range $-52° \leq \Phi \leq +52°$ for an orbit inclination of 28.5° with respect to the equator whereas Ψ is generally unrestricted.

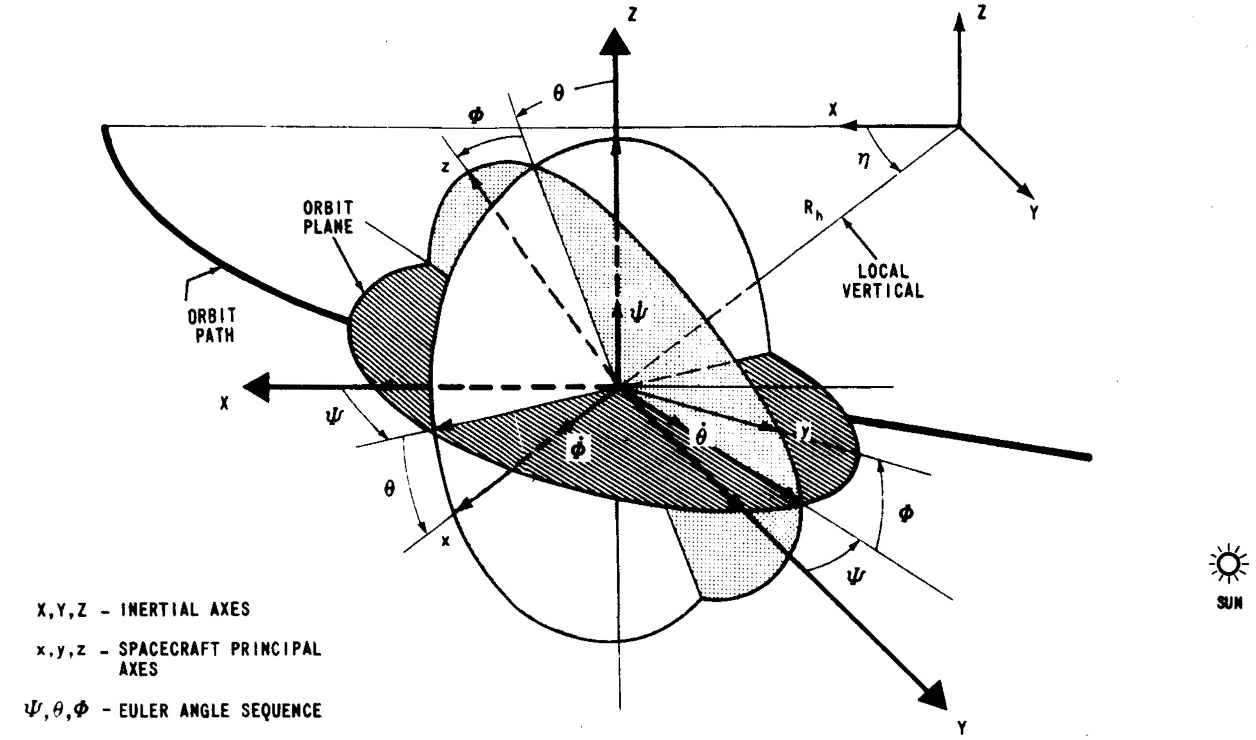

FIGURE I - SPACECRAFT ATTITUDE WITH RESPECT TO AN INERTIAL REFERENCE

2.2 Attitude Control

The spacecraft orientation described above for the ATM mission is also advantageous for the AAP 1/2 mission with fixed solar panels mounted normal to the y axis of the cluster configuration. In order to maintain a fixed attitude orientation however, the disturbance torque impulse $\int T_D dt$ acting on the spacecraft must be cancelled exactly. The basic equation describing the attitude behavior of the spacecraft is

$$\frac{d}{dt}(I\underline{\omega}) = I\underline{\dot{\omega}} + \underline{\omega} \times I\underline{\omega} = \underline{T}_D + \underline{T}_c \qquad (1)$$

which expresses the rate of change of spacecraft angular momentum in terms of the angular velocity

$$\underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\Phi & \cos\theta\ \sin\Phi \\ 0 & -\sin\Phi & \cos\theta\ \cos\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} \qquad (2)$$

and the principal axis inertia tensor

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad (3)$$

The term $\underline{T}_D$ represents the disturbance torque acting on the spacecraft including the gravity-gradient and aerodynamic torques and $\underline{T}_C$ is the control torque provided by an RCS or momentum exchange system.

In the AAP 1/2 mission CMG's are not available and the RCS fuel expenditure for a long duration inertial hold is prohibitive.* The Quasi-Inertial stabilization approach to be described in the next section leads to conditions under which the spacecraft is held approximately in the same orientation as in the ATM mode with comparatively low RCS fuel expenditure.

2.3 Quasi-Inertial Stabilization

In order to simplify the discussion it will be assumed that the spacecraft moves in a circular orbit, and initially that the spacecraft is symmetric so that $I_z = I_y$ and furthermore that $\theta(t) = \Phi(t) = 0$.** As described in Appendix A, Eq. (1) reduces to a single differential equation describing the motion of the vehicle in the orbit plane about the z axis.

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*Elrod, B. D., "Flight Attitude Alternatives for AAP 1/2," Bellcomm Technical Memorandum, TM-67-1022-2, April 21, 1967.

**The conditions required for achieving $\theta(t) = \Phi(t) = 0$ are discussed in Appendix A.

$$I_z \ddot{\Psi} = - T_{gm} \sin 2(\Psi - \eta) - \alpha T_{gm} \cos (\Psi - \eta) \qquad (4)$$

The term

$$T_{gm} = \frac{3\omega_o^2}{2} (I_z - I_x) \qquad (5)$$

represents the peak gravity-gradient torque and αT_{gm} represents the peak aerodynamic torque expressed as a fraction of T_{gm}.* The spacecraft orientation corresponding to this problem is shown in Figure 2.

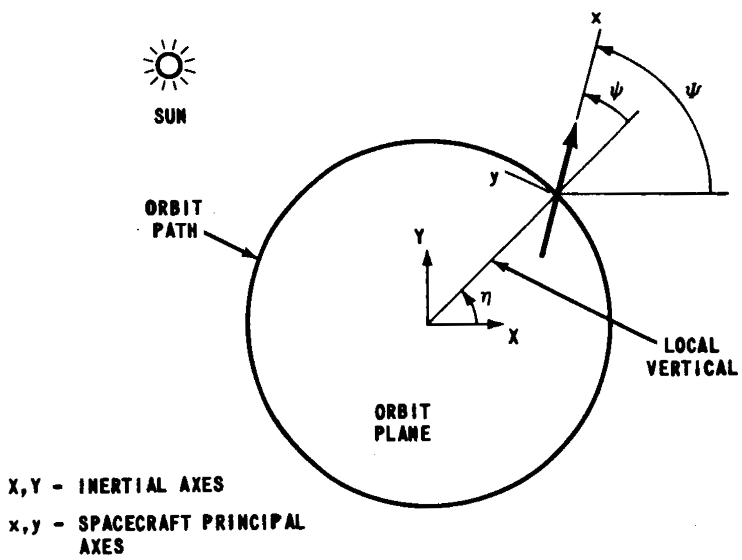

FIGURE 2 - SPACECRAFT ATTITUDE IN ORBIT PLANE

It is convenient for the analysis to consider the motion of the spacecraft in terms of the angular displacement ψ from the local vertical. From Figure 2 it follows that

$$\psi = \Psi - \eta = \Psi - \omega_o t \qquad (6)$$

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*In general, α may be positive or negative depending upon the relative location of the spacecraft CG and center of pressure. For convenience however, only $\alpha \geq 0$ will be considered, since the results are completely analogous for $\alpha < 0$. Normally $|\alpha| < 0.4$ at altitudes above 250 NM with the cluster configuration being considered for AAP 1/2.

where ω_o is the orbital angular velocity. Consequently Eq. (4) can be written as

$$\ddot{\psi} = - B \sin 2\psi - \alpha B \cos\psi \tag{7}$$

where

$$B \equiv T_{gm}/I_z = 3\omega_o^2 K_z/2 \tag{8}$$

and

$$K_z \equiv (I_z - I_x)/I_z \tag{9}$$

When the aerodynamic torque is negligible ($\alpha \approx 0$), Eq. (7) is similar to the differential equation obtained in describing the motion of a simple pendulum. A first integral of this differential equation can be obtained after multiplying by the integrating factor $2\dot{\psi}$ so that

$$\dot{\psi}^2 = 2B \cos^2\psi + C_1 \tag{10}$$

where

$$C_1 = \dot{\psi}_o^2 - 2B \cos^2\psi_o \tag{11}$$

Analogous to the pendulum problem, the motion of the spacecraft in the orbital plane is characterized by two possible modes. This can be illustrated by a phase plane plot of $\dot{\psi}$ vs ψ as shown in Figure 3 for various initial conditions. The closed trajectory (A) encircling the origin of the phase plane corresponds to oscillation about the local vertical. The outside trajectories (B and C) corresponding to continuous rotation with respect to the local vertical are analogous to a pendulum with sufficient initial velocity to continuously revolve about its pivot. The boundary on the phase plane dividing the two types of trajectories is known as the "separatrix". The equation for the separatrix can be obtained from Eqs. (10) and (11) by letting $\psi_o = 0$ and $\dot{\psi}_o = \pm \sqrt{2B}$ so that $C_1 = 0$ and

$$\dot{\psi}^2 \equiv \dot{\psi}_s^2 = 2B \cos^2\psi \tag{12}$$

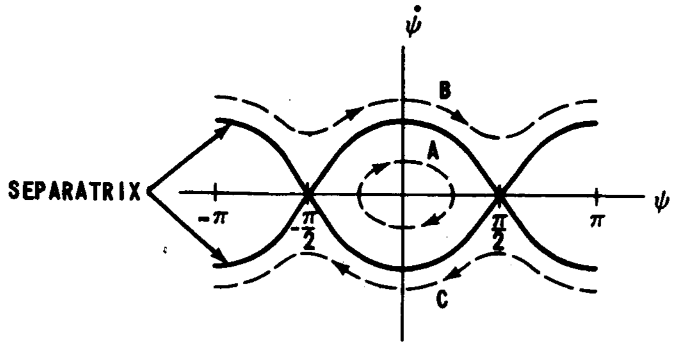

FIGURE 3 - PHASE PLANE TRAJECTORIES OF SPACECRAFT MOTION WITH RESPECT TO LOCAL VERTICAL

or

$$\dot{\psi}_s = \pm \sqrt{2B} \cos\psi \tag{13}$$

For trajectories outside the separatrix starting at $\psi_o = 0$ it follows that $|\dot{\psi}_o| > \sqrt{2B}$.

Evaluation of $\psi(t)$ for trajectories outside the separatrix can be obtained after integration of Eq. (10). This follows after writing Eq. (10) in the form

$$\dot{\psi}^2 = - 2B(1 - \cos^2\psi) + (C_1 + 2B)$$

$$= C_2 - 2B \sin^2\psi \tag{14}$$

where

$$C_2 = C_1 + 2B = \dot{\psi}_o^2 + 2B(1-\cos^2\psi_o) = \dot{\psi}_o^2 + 2B \sin^2\psi_o \tag{15}$$

Consequently,

$$\frac{d\psi}{dt} = \pm \sqrt{C_2} \sqrt{1 - k^2 \sin^2\psi} \tag{16}$$

where

$$k^2 \equiv 2B/C_2 \tag{17}$$

Integration of Eq. (16) yields*

$$\int_0^t dt = t = \pm \frac{1}{\sqrt{C_2}} \int_{\psi_o}^{\psi(t)} \frac{d\psi}{\sqrt{1 - k^2 \sin^2\psi}} = \pm \frac{1}{\sqrt{C_2}} \left[F\left(k,\psi(t)\right) - F(k,\psi_o)\right] \tag{18}$$

The function $F(k,\psi)$ is an elliptic integral of the first kind which is available in tabulated form as a function of the modulus k and argument ψ.** Two properties of the elliptic integral are

$$F(k,n\tfrac{\pi}{2}) = n\,F(k,\pi/2) = n\,K(k) \tag{19a}$$

and

$$F(k,n\pi+\beta) = nF(k,\pi) + F(k,\beta) = 2nK(k) + F(k,\beta) \tag{19b}$$

where β is an arbitrary angle and n is an integer ($n=0,\pm 1,\cdots$). The term $K(k) \equiv F(k,\pi/2)$ is the complete elliptic integral of the first kind.

The time τ required for a half revolution of the spacecraft with respect to the local vertical is obtained from Eq. (18) for $\psi(t) = \pi + \psi_o$. In view of Eq. (19b) this yields

$$\tau = \frac{1}{\sqrt{C_2}} \left\{F(k,\ \pi + \psi_o) - F(k,\psi_o)\right\} = \frac{2}{\sqrt{C_2}}\,K(k) \tag{20}$$

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*Proper interpretation of the signs ($\pm$) is important here since only $t>0$ is of interest.

**Jahnke & Emde, "Tables of Functions", 4th Ed., Dover Publications, New York, N. Y., 1945, pp. 52-89.

Also as a result of Eq. (19), the time interval for each subsequent half interval is identical. Consequently the time interval for a complete revolution is

$$T = \frac{1}{\sqrt{C_2}} \left\{ F(k, 2\pi + \psi_o) - F(k, \psi_o) \right\} = \frac{4}{\sqrt{C_2}} K(k) \tag{21}$$

The general trend in $\psi(t)$ for the two trajectories (B and C) outside the separatrix in Figure 3 is shown in Figure 4 for $\psi_o = 0$. As a result of the property of the elliptic integral expressed in Eq. (19), the deviations in $\psi(t)$ from a linear component, $\psi_{av}(t)$, are periodic. The linear component for the curve C can be expressed as

$$\psi_{av}(t) = -\frac{2\pi}{T} t = \dot{\psi}_{av} t \tag{22}$$

where

$$\dot{\psi}_{av} = -\frac{2\pi}{T} = -\frac{\pi}{2} \frac{\sqrt{C_2}}{K(k)} \tag{23}$$

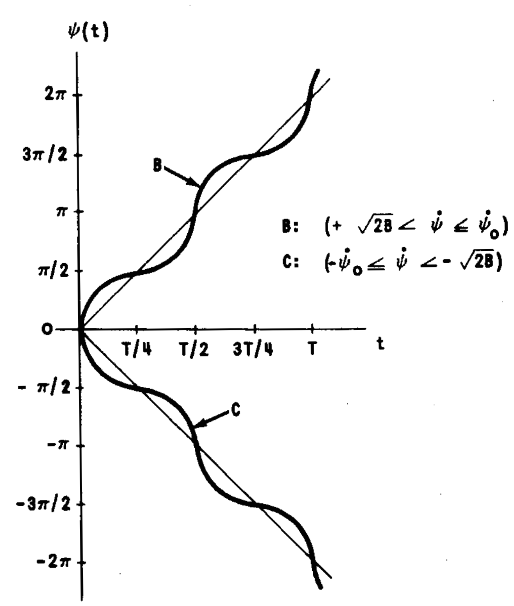

FIGURE 4 - DEVIATION OF SPACECRAFT FROM LOCAL VERTICAL

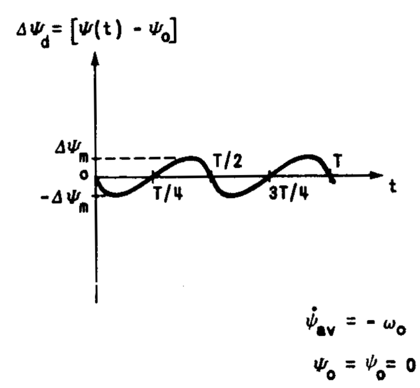

FIGURE 5 - VARIATION IN $\psi(t)$ FROM NOMINAL

The basic idea involved in the Quasi-Inertial stabilization scheme concerns the choice of $\dot{\psi}_{av}$. In view of Eq. (6) it follows that

$$\dot{\Psi} = \dot{\psi} + \omega_o \tag{24}$$

By appropriate selection of the initial angular velocity $\dot{\psi}_o$ such that $\dot{\psi}_{av} = -\omega_o$, the displacement of the spacecraft from an initial orientation (Ψ_o) normal to the sun line will be periodic with a maximum amplitude equal to the maximum deviation of $\psi(t)$ from the linear component $\psi_{av}(t)$. A plot of $\Psi(t) - \Psi_o$ corresponding to trajectory C in Figures 3 and 4 is shown in Figure 5 for $\dot{\psi}_{av} = -\omega_o$ and $\Psi_o = \psi_o = 0$.*

The foregoing analysis will be extended to include the effect of aerodynamic torque ($\alpha > 0$). The basic concept still involves an appropriate angular rate initialization ($\dot{\psi}_o$) such that $\dot{\psi}_{av} = -\omega_o$. The peak angular displacement $\Delta\Psi_m$ of the spacecraft from the nominal orientation will generally vary with α. In the next section the initial angular rate and the resultant peak displacement $\Delta\Psi_m$ are evaluated for several levels of aerodynamic torque.

3.0 DETERMINATION OF INITIALIZATION REQUIREMENTS

To establish the Quasi-Inertial mode the initial conditions ($\dot{\psi}_o$, ψ_o) must result in an average angular rate of the spacecraft which is just opposite to the orbital rate. In terms of the phase plane ($\dot{\psi}$ vs ψ) in Figure 3, the required initial conditions must lie somewhere on trajectory C if this is the trajectory with $\dot{\psi}_{av} = -\omega_o$. In the next two sections the required initial rate $\dot{\psi}_o$ with $\alpha = 0$ and $\alpha \neq 0$ will be evaluated at either $\psi_o = 0$ or $\psi_o = \pm\ \pi/2$, since these points correspond to convenient reference position in the orbit.** Also the peak angular

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*The choice of $\Psi_o = \psi_o = 0$ here is for convenience. In general $\Psi_o = \psi_o$ need not be zero.

**The initial points $\psi_o = 0$ and $\psi_o = \pm\ \pi/2$ are of interest since they correspond to the local vertical and local horizontal orientation of the spacecraft.

displacement $\Delta\Psi_m$ analogous to Figure 5 will be determined. In the following section the RCS fuel requirements for initialization are discussed and finally the effect of spacecraft asymmetry ($I_y \neq I_z$) on the Quasi-Inertial mode is considered.

3.1 Initial Conditions ($\alpha \neq 0$)

When the aerodynamic torque is negligible, the motion of the spacecraft in the orbit plane is described by Eq. (18) and either Eq. (14) or (10). The condition for the Quasi-Inertial mode is that $\dot{\psi}_{av} = -\omega_o$ which implies from Eq. (23) that

$$\dot{\psi}_{av} = -\frac{\pi}{2}\frac{\sqrt{C_2}}{K(k)} = -\omega_o \qquad (25)$$

In view of Eq. (17) it follows that

$$\sqrt{C_2} = \sqrt{2B}/k \qquad (26)$$

Consequently, Eq. (25) can be written as

$$\frac{\pi}{2}\lambda = k\,K(k) \qquad (27)$$

where

$$\lambda \equiv \sqrt{2B}/\omega_o = \sqrt{3K_z} = \sqrt{3(I_z - I_x)/I_z} \qquad (28)$$

follows from Eqs. (8) and (9). Upon evaluation of λ for a particular spacecraft configuration the transcendental function $k\,K(k)$ can be evaluated numerically to obtain $K(k)$ and the modulus k.

The initial angular rate $\dot{\psi}_o$ can be obtained from Eqs. (14), (26) and (28) which yield

$$\dot{\psi}_o^2 = C_2 - 2B\sin^2\psi_o = (\lambda/k)^2[1-k^2\sin^2\psi_o]\omega_o^2 \qquad (29)$$

so that

$$\dot{\psi}_o = -(\lambda/k)\sqrt{1 - k^2 \sin^2\psi_o}\;\omega_o \tag{30}$$

For $\psi_o = 0$ the initial rate is

$$\dot{\psi}_o \equiv \dot{\psi}_{oA} = -(\lambda/k)\omega_o \tag{31}$$

For $\psi_o = \pm\,\pi/2$ the initial rate is

$$\dot{\psi}_o \equiv \dot{\psi}_{oB} = -(\lambda/k)\sqrt{1-k^2}\;\omega_o \tag{32}$$

The initial points A and B are indicated on the phase plane shown in Figure 6.

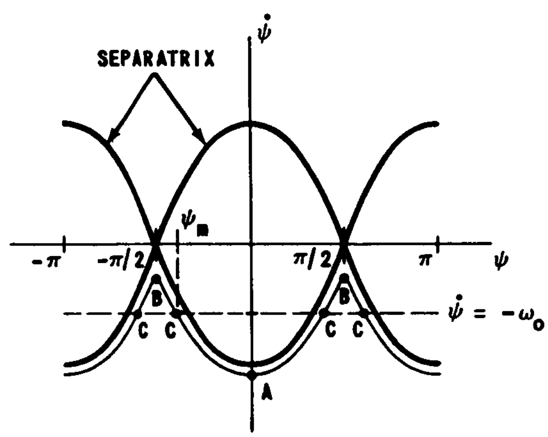

FIGURE 6 - PHASE PLANE TRAJECTORY IN QUASI-INERTIAL MODE

With $\dot{\psi}_{av} = -\omega_o$ the deviation of the spacecraft from the nominal orientation is periodic as illustrated in Figure 5. The maximum deviation $\Delta\Psi_m$ occurs when $\dot{\Psi} = 0$ or in view of Eq. (24), whenever $\dot{\psi} = -\omega_o$. This is indicated by point C in Figure 6. The corresponding value of $\psi = \psi_m$ can be obtained from Eqs. (14), (26) and (28), which yield

$$\dot{\psi}^2 = \omega_o^2 = C_2 - 2B \sin^2\psi_m = (2B/\omega_o^2)\ [1/k^2 - \sin^2\psi_m]\omega_o^2$$

$$= \lambda^2(1/k^2 - \sin^2\psi_m)\omega_o^2 \qquad (33)$$

or

$$\sin\psi_m = - \sqrt{1/k^2 - 1/\lambda^2} \qquad (34)$$

With this result and Eqs. (6), (18), (26) and (28) the maximum deviation of $\Psi(t)$ from $\Psi_o = \psi_o = 0$ (for convenience) becomes*

$$\Delta\Psi_m = \psi_m + \omega_o t_m = \psi_m - \frac{\omega_o}{\sqrt{C_2}} F(k,\psi_m) = \psi_m - \frac{k}{\lambda} F(k,\psi_m) \qquad (35)$$

All results in the foregoing analysis depend on the quantity $\lambda = \sqrt{3K_z}$ which is a physical parameter characterizing the spacecraft configuration. In general K_z is limited to the range $0 \leq K_z \leq 1$. For a uniform spherical spacecraft, $K_z = 0$ whereas $K_z \approx 1$ for long cylindrical spacecraft. The maximum deviation Ψ_m as well as the initial conditions corresponding to points A and B on the phase plane trajectory in Figure 6 are tabulated in Table I.

TABLE I - SUMMARY OF INITIAL ANGULAR RATE CONDITIONS & INERTIAL ATTITUDE DEVIATIONS

K_z	$\lambda = 3K_z$	k	$\dot{\psi}_{oA}/\omega_o$	$\dot{\psi}_{oB}/\omega_o$	ψ_m	$\Delta\Psi_m$
1.0	1.732	0.96873	-1.788	-0.444	-58.8°	18.8°
0.333	1.0	0.79310	-1.261	-0.768	-50.2°	7.0°
0	0	0**	-1	-1	0**	0**

*See Footnote * on page 8.

**In evaluating Eq. (27) as $\lambda \to 0$ it turns out that $K(k) \to \pi/2$ and $k \to \lambda$. As a result of Eq. (34), $\psi_m \to 0$, which implies that $\Psi_m \to 0$ from Eq. (35).

The results for $\Delta\Psi_m$ in Table I are to be expected since the gravity-gradient torque has a decreasing effect as the spacecraft configuration becomes more spherical. The Orbital Assembly configuration to be used in the AAP 1/2 mission is long and cylindrical such that $0.9 < K_z < 1.0$. Consequently, a maximum deviation in the neighborhood of $\Delta\Psi_m = 18°$ from the nominal would be expected if the aerodynamic torque was negligible.

3.2 Initial Conditions ($\alpha \neq 0$)

When aerodynamic torque is not negligible ($\alpha \neq 0$), solution of the differential equation

$$\ddot{\psi} = - B \sin 2\psi - \alpha B \cos\psi \tag{36}$$

describing the motion of the spacecraft in the orbital plane is more complicated. In this section a solution of Eq. (36) is obtained and used to obtain initial conditions for establishing the Quasi-Inertial mode.

A first integration of Eq. (36) can be obtained after multiplication by the integrating factor $2\dot{\psi}$ so that

$$\begin{aligned} \dot{\psi}^2 &= -2B \sin^2\psi - 2\alpha B \sin\psi + C \\ &= -2B [\sin^2\psi + \alpha \sin\psi + \alpha^2/4] + (C + B\,\alpha^2/2) \\ &= C_3 - 2B(\sin\psi + \alpha/2)^2 \end{aligned} \tag{37}$$

where

$$C_3 = \dot{\psi}_o^2 + 2B(\sin\psi_o + \alpha/2)^2 \tag{38}$$

When $\alpha = 0$, these equations reduce to Eqs. (14) and (15).

The motion of the spacecraft is again characterized by two possible modes as illustrated in the phase plane plot of $\dot{\psi}$ vs ψ as shown in Figure 7 for various initial conditions and $\alpha > 0$. The boundary or separatrix dividing the two types of trajectories can be obtained from Eqs. (37) and (38) be letting $\psi_o \equiv \psi_\alpha = -\sin^{-1}(\alpha/2)$ and $\dot{\psi}_o = \pm \sqrt{2B}\,(1 + \alpha/2)$. This yields

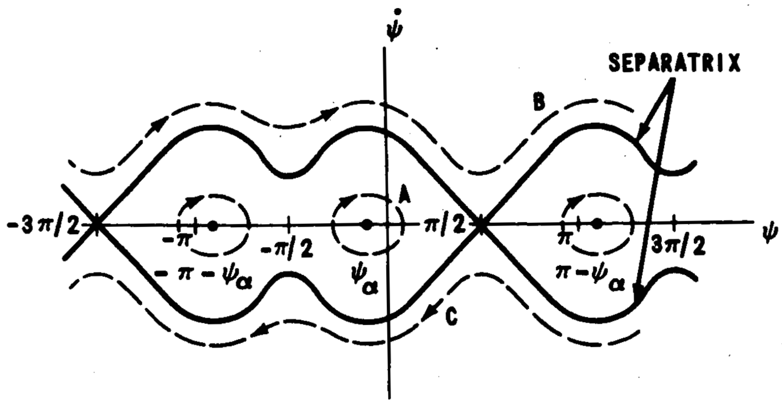

FIGURE 7 - PHASE PLANE TRAJECTORIES OF SPACECRAFT MOTION WITH RESPECT TO LOCAL VERTICAL (α>0)

$$\dot{\psi}^2 \equiv \dot{\psi}_s^2 = 2B(1 + \alpha/2)^2 - 2B(\sin\psi + \alpha/2)^2$$

$$= 2B[1 + \alpha + \alpha^2/4 - \sin^2\psi - \alpha\sin\psi - \alpha^2/4]$$

$$= 2B[\cos^2\psi + \alpha(1 - \sin\psi)] \qquad (39)$$

or

$$\dot{\psi}_s = \pm\sqrt{2B}\,[\cos^2\psi + \alpha(1 - \sin\psi)]^{1/2} \qquad (40)$$

which is equivalent to Eq. (13) when $\alpha = 0$. The angular velocities $\dot{\psi}_{s\ max}$ and $\dot{\psi}_{s\ min}$ corresponding to the maximum and local minimum points on the separatrix can be evaluated from Eq. (40) for $\psi_s = n\pi + (-1)^n\psi_\alpha$ and $\psi_s = 2n\pi - \pi/2$ respectively, where $n = 0, \pm1, \pm2,\cdots$ and $\psi_\alpha = -\sin^{-1}(\alpha/2)$. The result is

$$\dot{\psi}_{s\ max} = \pm\sqrt{2B}\,[\cos^2\psi_\alpha + \alpha(1 - \sin\psi_\alpha)]^{1/2}$$

$$= \pm\lambda\omega_o\{[1 - (\alpha/2)^2] + \alpha(1 + \alpha/2)\}^{1/2}$$

$$= \pm(1 + \alpha/2)\lambda\omega_o \qquad (41)$$

and

$$\dot{\psi}_{s\ min} = \pm \sqrt{2B}\,[2\alpha]^{1/2} = \pm \sqrt{2\alpha}\,\lambda\omega_o \qquad (42)$$

where $\lambda = \sqrt{3K_z} = \sqrt{2B}/\omega_o$ as defined in Eq. (28).

For trajectories inside the separatrix the effect of the aerodynamic torque is to shift the points of equilibrium* so that the spacecraft no longer oscillates symmetrically about the local vertical. Outside the separatrix the trajectories B and C, as before, correspond to continuous rotation with respect to the local vertical.

The second integration of Eq. (36) to obtain $\psi(t)$ for trajectories outside the separatrix again leads to an elliptic integral. From Eq. (37) it follows that

$$\frac{d\psi}{dt} = \pm \sqrt{C_3}\,\sqrt{1 - k_a^2\,(\sin\psi + \alpha/2)^2} \qquad (43)$$

where

$$k_a^2 \equiv 2B/C_3 \qquad (44)$$

Integration of Eq. (43) yields

$$\int_0^t dt = t = \pm \frac{1}{\sqrt{C_3}} \int_{\psi_o}^{\psi(t)} \frac{d\psi}{\sqrt{1 - k_a^2\,(\sin\psi + \alpha/2)^2}} \qquad (45)$$

which is an elliptic integral, although not in standard form unless $\alpha = 0$. The reduction of the integral to standard form is

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*An equilibrium point is a condition of zero angular velocity ($\dot{\psi} = \dot{\psi}_o = 0$) for which the angular orientation of the spacecraft remains fixed with respect to local vertical, since the gravity-gradient and aerodynamic torques are equal and opposite.

carried out in Appendix B. The result is*

$$t = \pm \frac{1}{\sqrt{C_3}} \left\{ \frac{1}{k_a m} \int_{\sigma_o}^{\sigma(t)} \frac{d\sigma}{\sqrt{1 - k_e^2 \sin^2\sigma}} \right\}$$

$$= \pm \frac{1}{\sqrt{C_3}\, k_a m} \left\{ F[k_e, \sigma(t)] - F(k_e, \sigma_o) \right\} \qquad (46)$$

The relationship between $\psi(t)$ and $\sigma(t)$ as given in Eq. (B-37) is

$$\sin \sigma(t) = \frac{\sin \psi(t) + n_e}{1 + n_e \sin \psi(t)} \qquad (47)$$

The parameters m, k_e and n_e are all functions of k_a and α. They are given in Appendix B in Eqs. (B-42), (B-43) and (B-44) respectively. When $\alpha \to 0$, the expression for t reduces to Eq. (18), since $n_e \to 0$, $m \to 1/k_a$, $k_e \to k_a \to k$ and $C_3 \to C_2$.

The time interval for one revolution of the spacecraft with respect to local vertical can be expressed as

$$T = \frac{1}{\sqrt{C_3}\, k_a m} \left[F(k_e, 2\pi + \sigma_o) - F(k_e, \sigma_o) \right]$$

$$= \frac{4}{\omega_o \lambda m} F(k_e, \pi/2) = \frac{4}{\omega_o \lambda m} K(k_e) \qquad (48)$$

where

$$\sqrt{C_3}\, k_a = (\sqrt{2B}/\omega_o)\omega_o = \lambda \omega_o \qquad (49)$$

---

*See Footnote * on page 8.

due to Eqs. (28) and (44). The solution for $\psi(t)$ is similar to the result shown in Figure 4 for $\alpha = 0$. The variation about the average component is a function of α as well as the spacecraft configuration parameter λ.

The average component of $\psi(t)$ for trajectory C in Figure 7 is given by

$$\psi_{av}(t) = \dot{\psi}_{av} t = - \frac{2\pi}{T} t \tag{50}$$

For the Quasi-Inertial mode it is necessary that

$$\dot{\psi}_{av} = - \frac{2\pi}{T} = - \frac{\pi}{2} \frac{\omega_o \lambda m}{K(k_e)} = - \omega_o \tag{51}$$

or

$$\frac{\pi}{2} \lambda = \frac{K(k_e)}{m} = \frac{K[k_e(k_a,\alpha)]}{m(k_a,\alpha)} \tag{52}$$

Upon specification of α and λ the value of k_a which satisfies this transcendental equation can be determined numerically.

The initial angular rate $\dot{\psi}_o$ for the Quasi-Inertial mode can be obtained from Eqs. (37), (44) and (28) which yield

$$\dot{\psi}_o^2 = C_3 - 2B(\sin\psi + \alpha/2)^2 = (2B/\omega_o^2)[1/k_a^2 - (\sin\psi_o + \alpha/2)^2]\omega_o^2$$

$$= (\lambda^2/k_a^2)[1 - k_a^2(\sin\psi_o + \tfrac{\alpha}{2})^2]\omega_o^2 \tag{53}$$

so that*

$$\dot{\psi}_o = - (\lambda/k_a) \sqrt{1 - k_a^2(\sin\psi_o + \alpha/2)^2}\ \omega_o \tag{54}$$

---

*If the inertial initial rate is desired it is only necessary to add ω_o since $\dot{\Psi} = \dot{\psi}_o + \omega_o$.

For $\psi_o = +\pi/2$ and $\psi_o = -\pi/2$ the required initial rates are respectively

$$\dot{\psi}_o = \dot{\psi}_{oA} = -(\lambda/k_a)\sqrt{1 - k_a^2(1 + \alpha/2)^2}\,\omega_o \tag{55}$$

and

$$\dot{\psi}_o = \dot{\psi}_{oB} = -(\lambda/k_a)\sqrt{1 - k_a^2(1 - \alpha/2)^2}\,\omega_o \tag{56}$$

The initial points A and B are indicated on the phase plane in Figure 8.

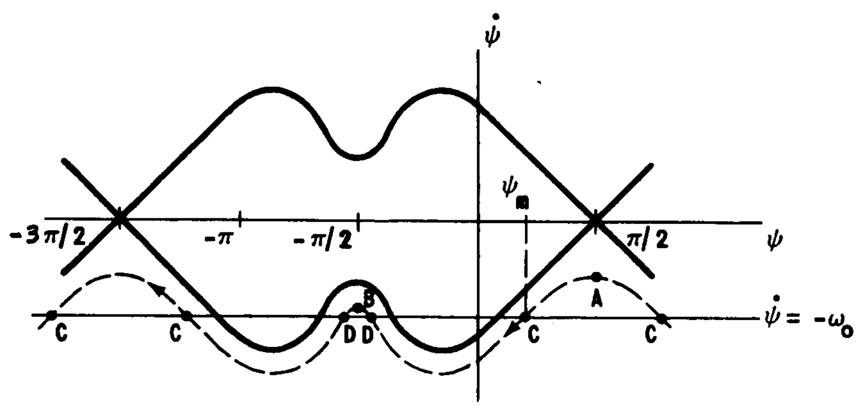

FIGURE 8 - PHASE PLANE TRAJECTORY IN QUASI INERTIAL MODE

The inertial attitude of the spacecraft as expressed in Eq. (6) is

$$\Psi = \psi + \omega_o t \tag{57}$$

where $\Psi_o = \psi_o$ is the nominal orientation. For $\dot{\psi}_{av} = -\omega_o$ the deviation from the nominal is periodic. When $\alpha = 0$, the period is half the orbital period T and the deviation in each successive quarter period is symmetric with equal maximum deviations $\Delta\Psi_m$ as given by Eq. (35) and illustrated in Figure 5. When $\alpha \neq 0$, the peak deviations are not necessarily equal in successive quarter periods. Due to asymmetry as shown on the phase plane of Figure 8 the departure of the trajectory from the line $\dot{\psi} = -\omega_o$ is less at $\psi = -\pi/2$ than at $\psi = +\pi/2$. Consequently, the maximum deviation $\Delta\Psi_m$ which occurs when $\dot{\Psi} = 0$ or equivalently whenever $\dot{\psi} = -\omega_o$, will correspond to a point (C) near $\psi = \pi/2, -3\pi/2,$ etc. in Figure 8 rather than a point (D) near $\psi = -\pi/2, -5\pi/2,$ etc. In fact for sufficiently large α, the trajectory will no longer cross the line $\dot{\psi}_o = -\omega_o$ near $\psi = -\pi/2$,* so that only two maxima will result during each orbit. The value of $\psi \equiv \psi_m$ corresponding to the point C between $\psi = 0$ and $\psi = \pi/2$ can be obtained from Eqs. (37), (44) and (28) which yield

$$\dot{\psi}_C^2 = \omega_o^2 = C_3 - 2B(\sin\psi_m + \alpha/2)^2$$

$$= \lambda^2[1/k_a^2 - (\sin\psi_m + \alpha/2)^2]\omega_o^2 \qquad (58)$$

or

$$\sin\psi_m = + \sqrt{1/k_a^2 - 1/\lambda^2} - \alpha/2 \qquad (59)$$

The maximum deviation $\Delta\Psi_m$ from the nominal orientation can be determined from Eqs. (57), (46), (44) and (28). The result is

$$\Delta\Psi_m = \Psi_m - \Psi_o = (\psi_m + \omega_o t_m) - \psi_o = (\psi_m - \psi_o)$$

$$+ \omega_o \left\{ \frac{-1}{\sqrt{2B}m} [F(k_e,\sigma_m) - F(k_e,\sigma_o)] \right\}$$

$$= (\psi_m - \psi_o) + \frac{1}{\lambda m} [F(k_e,\sigma_o) - F(k_e,\sigma_m)] \qquad (60)$$

---

*On the separatrix the angular rate corresponding to $\psi = -\pi/2$ is given by Eq. (42) as $\dot{\psi}_{s\ min} = -\sqrt{2\alpha}\lambda\omega_o$. Consequently, for $\alpha > 1/2\lambda^2 = 1/6K_z$, no intersections with the line $\dot{\psi}_o = -\omega_o$ are possible near $\psi = -\pi/2, -5\pi/2,$ etc.

where

$$\sin\sigma_m = \frac{\sin\psi_m + n_e}{1 + n_e \sin\psi_m} \tag{61}$$

In general, both the initial angular rate $\dot{\psi}_o$ and the maximum deviation $\Psi_m - \Psi_o = \Delta\Psi_m$ vary with α as well as the spacecraft configuration parameter $\lambda = \sqrt{3K_z}$. This variation is indicated in Table II where $\dot{\psi}_o/\omega_o$, $\dot{\Psi}_o/\omega_o$, $\Delta\Psi_m$ and ψ_m are tabulated for various values of α.

TABLE II - SUMMARY OF INITIAL ANGULAR RATE CONDITIONS & MAXIMUM INERTIAL ATTITUDE DEVIATION VS α

Data \ α	$\alpha = 0$	$\alpha = 0.03$	$\alpha = 0.12$	$\alpha = 0.4$
$\dot{\psi}_o/\omega_o$	-0.4675	-0.3856	-0.2427	-0.1120
$\dot{\Psi}_o/\omega_o$	0.5325	0.6144	0.7573	0.8880
$\Delta\Psi_m$	17.8°	23.7°	38.0°	59.1°
ψ_m	58.13°	56.86°	55.86°	57.46°
Parameters				
k_a	0.9632	0.9608	0.9347	0.8320
k_e	0.9632	0.9637	0.9660	0.9753
m	1.0382	1.0389	1.0524	1.1101
n_e	0	-0.1870	-0.5580	-0.8611

These data are the results of calculations based on the foregoing analysis with $\Psi_o = \psi_o = 90°$ and $K_z = 0.9346$.* The initial

*The configuration parameter $K_z = 0.9346$ used here is typical for the Orbital Assembly configuration being considered for the AAP 1/2 mission.

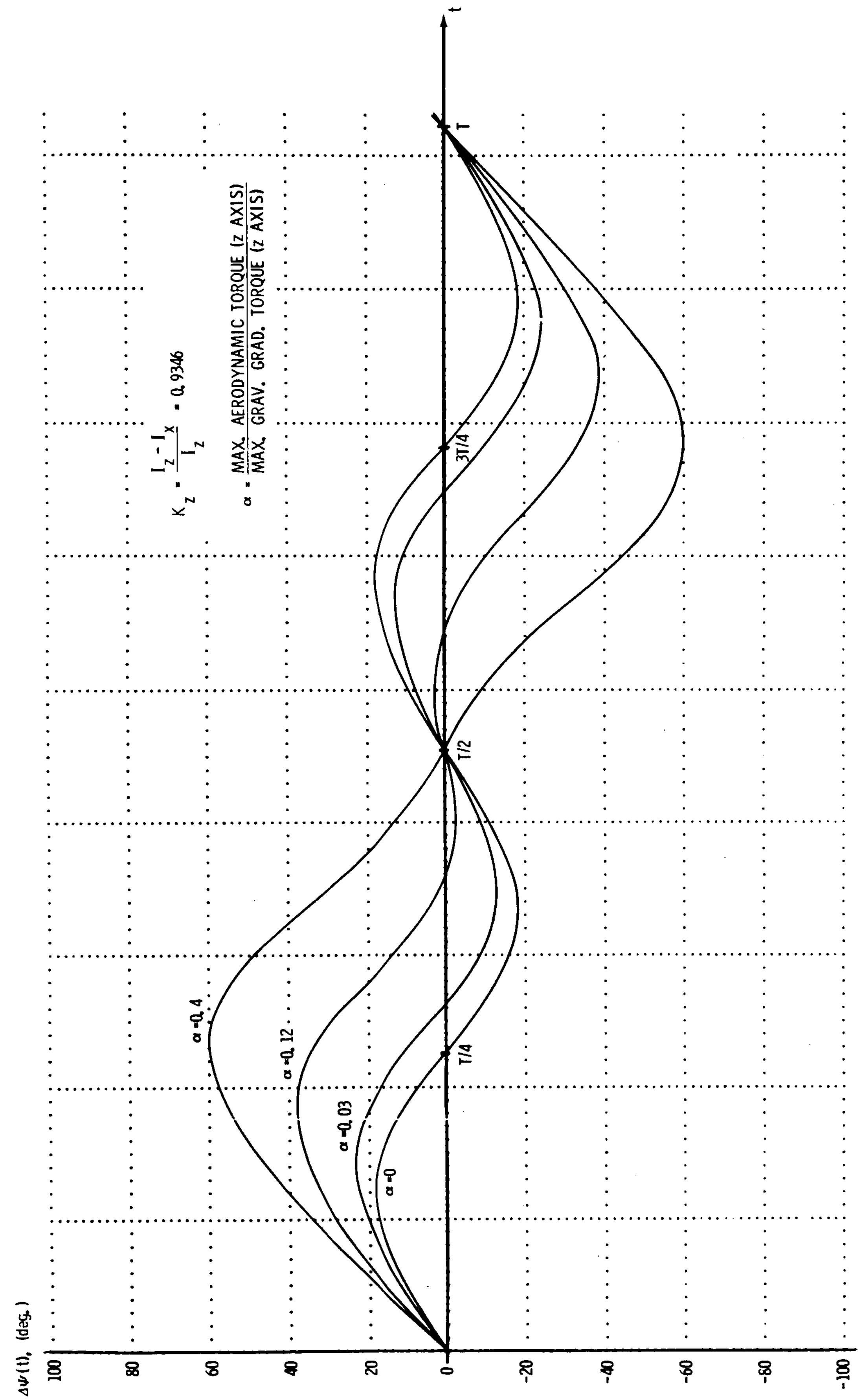

FIGURE 9 - DEVIATION FROM INERTIAL ORIENTATION IN QUASI-INERTIAL MODE

conditions, $\dot{\psi}_o$ and $\psi_o = 90°$, were used in a digital computer solution of Eq. (36) to obtain a plot of $\Delta\Psi(t)$ as shown in Figure 9 for $\alpha = 0$, 0.03, $\alpha = 0.12$ and $\alpha = 0.4$. The maximum deviations $\Delta\Psi_m$ compare favorably with the calculations listed in Table II.

3.3 RCS Fuel Requirements for Initialization

In order to establish the Quasi-Inertial mode exactly it is generally necessary to achieve a set of initial conditions $(\dot{\psi}_o, \psi_o)$ such that the average angular rate of the spacecraft is just opposite to the orbital rate ω_o. If $\dot{\psi}_{av} = - \rho\omega_o$ with $\rho \neq 1$, the deviation $\Delta\Psi$ of the spacecraft from the desired inertial orientation Ψ_o contains a linear component

$$\Psi_\ell(t) = - \rho\omega_o t + \omega_o t = (1 - \rho)\omega_o t \qquad (62)$$

in addition to a periodic variation as described previously. The accumulated deviation after one orbit without an intermediate re-initialization is

$$\Psi_\ell(T) = (1 - \rho)\omega_o T = (1 - \rho)2\pi \qquad (63)$$

Thus, a 10% rate initialization error* produces an accumulated linear deviation of 36° after one orbit or 18° after a half orbit. Re-initialization is required when $\Psi_\ell(t)$ exceeds a tolerable level.

Several methods for accomplishing the initialization operation can undoubtedly be devised. A complete examination of various initialization schemes and corresponding control system requirements is beyond the scope of this report. However, in order to provide some indication of the RCS fuel requirements for initialization a scheme based only on angular rate initialization once per orbit will be described.

---

*Since the orbital angular rate ω_o is in the order of 0.064 ± 0.002 deg/sec for 200-300 NM orbits, rate errors less than 10% imply a spacecraft rate control capability of at least $\pm$ 0.003 deg/sec.

In Figure 10 the phase plane trajectories corresponding to three different values of $\dot{\psi}_{av}$ are shown.* The center

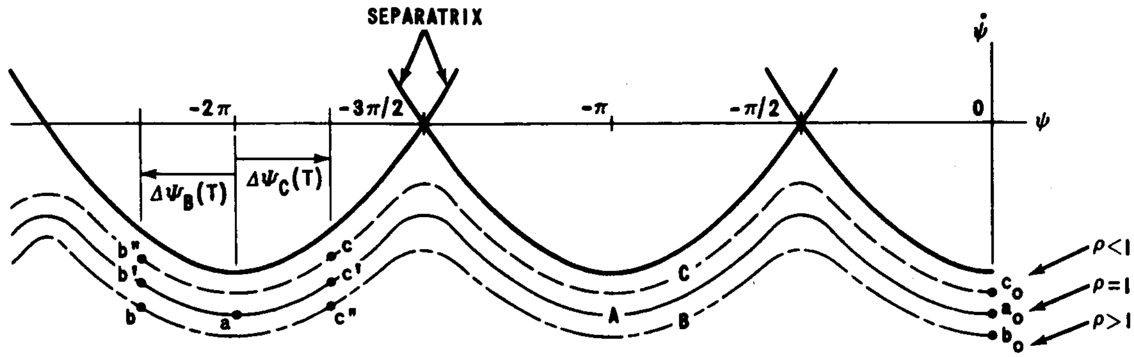

FIGURE 10 - PHASE PLANE TRAJECTORIES OF SPACECRAFT MOTION WITH RESPECT TO LOCAL VERTICAL

trajectory (A) is the desired trajectory with $\rho = 1$. From an initial condition corresponding to points a_o, b_o, or c_o in Figure 10 the spacecraft would move along the respective trajectories to points a, b, or c after one orbit. Since

$$\Psi(T) = \psi(T) + \omega_o T = \psi(T) + 2\pi \tag{64}$$

the angular deviation associated with trajectories B and C are $\Delta\Psi_B$ and $\Delta\Psi_C$ as indicated in Figure 10. The same information is indicated in Figure 11 where $\Delta\Psi(t)$ is plotted for the three cases.

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*In order to simplify the discussion only the trajectories for $\alpha = 0$ are considered, since the results are similar when aerodynamic torque is not negligible ($\alpha \neq 0$).

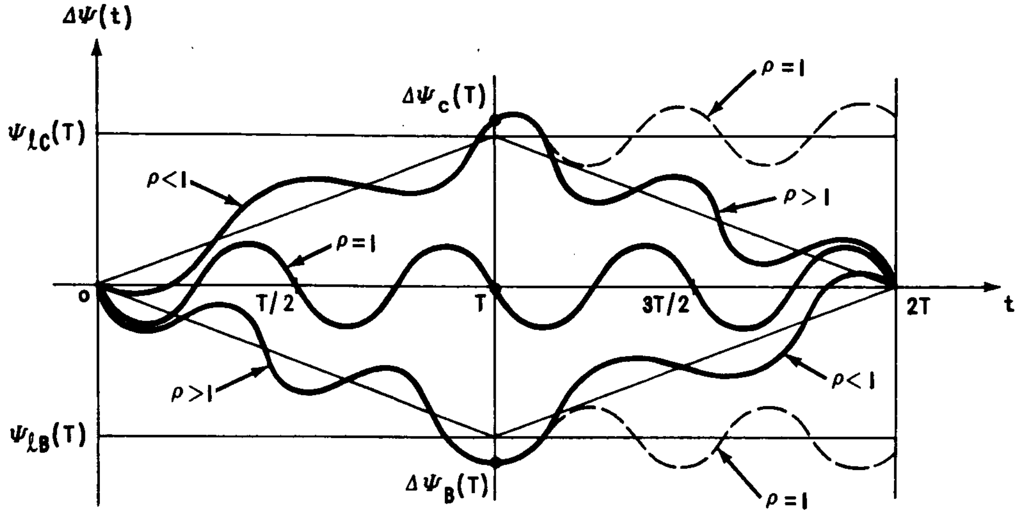

FIGURE 11 - VARIATION IN $\Psi(t)$ FROM NOMINAL

If the average angular rate of the spacecraft on trajectories B or C is re-initialized to bring point b to b' or c to c' respectively, the angular deviation $\Delta\Psi(t)$ would again vary periodically, but about $\Psi_\ell(T)$ as shown by the dashed curves in Figure 11, since $\dot{\psi}_{av} = - \omega_o$. It would be better however, to re-initialize from point b to b" and from c to c" in order to reduce the linear deviation from $\Psi_\ell(T)$ to zero during the next orbital period.* The corresponding deviation $\Delta\Psi(t)$ is indicated by the solid curves in Figure 11 for $T < t < 2T$. The RCS fuel requirement would then be based on an angular rate correction $\Delta\dot{\Psi}_c$ of twice the original rate error, specifically

---

*At $t = 2T$ another rate correction could be applied to achieve $\dot{\psi}_{av} = - \omega_o$. In general the companion trajectory of B or C used in reducing $\Psi_\ell(t)$ to zero in exactly one orbit as shown need not coincide with trajectory C or B respectively. For purposes of illustration however, it is assumed that trajectories B and C are companion trajectories. It should be noted that other companion trajectories outside the band defined by B and C would be associated with initialization schemes based on rate correction more frequently than once per orbit.

$$\Delta\dot{\Psi}_c = 2\Delta\dot{\Psi}_c = 2(1 - \rho)\omega_o \tag{65}$$

Consequently for a rate initialization error of 10% the magnitude of the desired rate correction is $|\Delta\dot{\Psi}_c| = 0.2\omega_o$. The corresponding change in spacecraft momentum must be

$$\Delta H = I_z|\Delta\dot{\Psi}_c| = 0.2I_z\omega_o \tag{66}$$

Since the required control torque impulse $\int T_c dt$ must equal ΔH, the RCS fuel consumption when firing two jets as a couple is

$$W_F = 2\,\frac{\int T_c dt}{I_{sp}L} = \frac{2\Delta H}{I_{sp}L} = \frac{0.4\ I_z\omega_o}{I_{sp}L} \tag{67}$$

where I_{sp} is the specific impulse of the RCS propellant and L is the distance between jets.

A worst case estimate of the total RCS fuel requirement for a long duration mission can be made assuming that on the average an angular rate correction equivalent to a 10% rate initialization error must be made every orbit. For $I_z = 2.0 \times 10^6$ slug-ft^2, $I_{sp} = 280$ sec, L = 12.8 ft. and $\omega_o = 1.11 \times 10^{-3}$ rad/sec for a 270 NM orbit, the RCS fuel consumption according to Eq. (67) is

$$W_F = \frac{0.4(2 \times 10^6)(1.11 \times 10^{-3})}{(280)\ (12.8)} = 0.25 \text{ lbs/orbit} \tag{68}$$

The RCS fuel consumption rate for initialization is then approximately 4 lbs/day or 120 lbs/month.

4.0 EFFECT OF SPACECRAFT ASYMMETRY ON QUASI-INERTIAL MODE

Throughout the foregoing analysis of the Quasi-Inertial mode the spacecraft was assumed to be symmetrical ($I_y = I_z$). The attitude control function then involved only a recurrent initialization operation for the purpose of sustaining the mode. When the spacecraft is not symmetrical ($I_y \neq I_z$), the differential equation describing, the motion of the principal axis of minimum moment of inertia (x axis) is modified slightly. Furthermore an additional attitude control requirement results, since the gravity-gradient torque about the x axis is no longer zero unless $\Phi_o = 0$.* In this section the modification to the differential equation is noted and the RCS fuel consumption associated with the x axis control requirement is evaluated.

According to Eqs. (A-13) and (A-21) in Appendix A the expressions for the equivalent disturbance torque T_{dx} and the differential equation describing the x axis motion in the orbital plane are respectively

$$T_{dx} = (T_{gmx}/3)\sin 2\Phi_o \; [(\dot{\Psi}/\omega_o)^2 + 3\sin^2(\Psi-\eta)] \tag{69}$$

and

$$\ddot{\Psi} = -B' \sin 2(\Psi-\eta) - \alpha' B' \cos(\Psi-\eta) \tag{70}$$

Here T_{gmx} is the peak gravity-gradient torque about the x axis and α' is related to the parameter α which is the ratio of peak aerodynamic torque to peak gravity-gradient torque about the z axis.

In the previous analysis it was convenient to consider the motion of the spacecraft in terms of the angular displacement ψ from the local vertical where

$$\psi = \Psi - \eta = \Psi - \omega_o t \tag{71}$$

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*This is clearly a special case since in general the out-of-orbit-plane solar pointing angle may be in the range $-52° < \Phi < +52°$ for an orbit inclination of 28.5° with respect to the equator.

Consequently Eqs. (69) and (70) can be written as

$$T_{dx} = (T_{gmx}/3)\sin 2\Phi_o \, [(\dot{\psi}/\omega_o + 1)^2 + 3\sin^2\psi] \qquad (72)$$

and

$$\ddot{\psi} = -B' \sin 2\psi - \alpha' B' \cos\psi \qquad (73)$$

The latter result differs from Eq. (36) in that the constants α' and B' both depend on Φ_o in addition to the inertia parameters as indicated in Eqs. (A-21) through (A-29). However, as also noted in Appendix A, the constants B' and α' do not differ significantly from the constants B and α in Eq. (36) when $I_z > I_y >> I_x$, as is true for the Orbital Assembly configuration. Consequently the results in Table II will not differ greatly when slight spacecraft asymmetry is taken into account.

The RCS fuel consumption per orbit associated with the control torque requirement on the x axis can be calculated from

$$W_F = 2\frac{\int_o^T T_{cx}dt}{I_{sp}L} = 2\frac{\int_o^T T_{dx}dt}{I_{sp}L} \qquad (74)$$

where $\int T_{cx}dt$ is the control torque impulse produced by the RCS thrusters, I_{sp} is the specific impulse of the RCS propellant and L is the distance between two RCS jets fired as a couple. The expression for T_{dx} in Eq. (72) involves both $\dot{\psi}$ and ψ. In view of Eqs. (37), (44) and (49), $\dot{\psi}$ can be obtained from

$$\dot{\psi}^2 = C_3 - 2B'(\sin\psi + \alpha'/2)^2$$

$$= (\lambda'/k_a')^2[1 - k_a'^2(\sin\psi + \alpha'/2)^2]\omega_o^2 \qquad (75)$$

or

$$d\psi/dt = \dot{\psi} = -(\lambda'/k_a') \sqrt{1 - k_a'^2 (\sin\psi + \alpha'/2)^2}\ \omega_o$$

$$= -(\lambda'/k_a')\ M(\psi)\omega_o \tag{76}$$

where

$$M(\psi) \equiv \sqrt{1 - k_a'^2 (\sin\psi + \alpha'/2)^2} \tag{77}$$

$$\lambda' \equiv \sqrt{2B'}/\omega_o \tag{78}$$

and

$$k_a' \equiv \sqrt{2B'/C_3} \tag{79}$$

After substituting Eq. (76) into Eq. (72) the expression for T_{dx} becomes

$$T_{dx} = (T_{gmx}/3)\sin 2\Phi_o\ N(\psi) \tag{80}$$

where

$$N(\psi) = [(\lambda'/k_a')^2\ M^2(\psi) - 2(\lambda'/k_a')\ M(\psi) + 1 + 3\sin^2\psi] \tag{81}$$

The required control torque impulse ${}_o\int^T T_{cx}dt$ is then given by

$$\int_o^T T_{cx}dt = \int_o^T T_{dx}dt = (T_{gmx}/3)\sin 2\Phi_o \int_o^{-2\pi} N(\psi)d\psi/\dot{\psi}$$

$$= (T_{gmx}/3)\sin 2\Phi_o\ [G/\omega_o(\lambda'/k_a')] \tag{82}$$

where $\psi_o = 0$ for convenience and

$$G \equiv -\int_0^{-2\pi} \frac{N(\psi)}{M(\psi)} d\psi = -\int_0^{-2\pi} [(\lambda'/k_a')^2 M(\psi) - 2(\lambda'/k_a) + 1/M(\psi) + 3\sin^2\psi/M(\psi)]d\psi \tag{83}$$

The three terms in G involving $M(\psi)$ lead to elliptic integrals of either the first or second kind, although not in a standard form unless $\alpha = 0$. When $\alpha = 0$, the result is*

$$G = 4\left\{(\lambda'/k_a')^2 E(k_a') - \frac{4\pi}{4} (\lambda'/k_a') + K(k_a') + (3/k_a'^2)[K(k_a') - E(k_a')]\right\} \tag{84}$$

where

$$\int_0^{-2\pi} d\psi/M(\psi) = F(k_a', -2\pi) = -4F(k_a', \pi/2) = -4K(k_a') \tag{85}$$

and

$$\int_0^{-2\pi} M(\psi)d\psi = E(k_a', -2\pi) = -4E(k_a', \pi/2) = -4E(k_a') \tag{86}$$

The complete elliptic integrals of the first and second kinds, $K(k_a')$ and $E(k_a')$ respectively, can be evaluated from tables** once the modulus k_a' is specified. Finally, after substitution of Eq. (84) into Eq. (82) the RCS fuel consumption per orbit (with $\alpha = 0$) can be expressed as

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*The last term in G can be expanded to the form

$$\sin^2\psi/M(\psi) = (1/k_a'^2)[1/M(\psi) - M(\psi)]$$

**Jahnke and Emde, op. cit.

$$W_F = 2\,\frac{\int_0^T T_{cx}dt}{I_{sp}L}$$

$$= \frac{8T_{gmx}|\sin 2\Phi_o|}{3\omega_o(\lambda'/k_a')I_{sp}L}\left\{(1+3/k_a'^2)K(k_a')+[(\lambda'/k_a')^2-3/k_a'^2]E(k_a')-\pi(\lambda'/k_a)\right\} \tag{87}$$

As a specific example the RCS fuel requirement will be evaluated for spacecraft asymmetry in the order of $(I_z - I_y) = 0.1 \times 10^6$ slug-ft^2. In particular the spacecraft inertias $I_z = 2.06 \times 10^6$, $I_y = 1.96 \times 10^6$ and $I_x = 0.135 \times 10^6$ are considered. As noted in Table A-1 of Appendix A, the constant B' is virtually unaffected by the asymmetry. Consequently λ' and k_a' are identical to the values of λ and k_a used in connection with Table II for $\alpha = 0$ and $I_z = I_y = 2.06 \times 10^6$ slug-ft^2. Thus, $(\lambda'/k_a') = 1.674$ and $k_a' = 0.963$ so that $E(k_a') = 1.08$ and $K(k_a') = 2.732$.* For $I_{sp} = 280$ sec., $L = 12.7$ ft. and $\omega_o = 1.11 \times 10^{-3}$ rad/sec. (270 NM circular orbit) the RCS fuel consumption is then

$$W_F = 0.44|\sin 2\Phi_o| \text{ lbs/orbit}$$

$$= 6.65|\sin 2\Phi_o| \text{ lbs/day} \tag{88}$$

Since the fuel consumption varies from day to day due to the variation in Φ_o, the total fuel requirement for a particular mission duration should be based on the average value of $|\sin 2\Phi_o|$.** If the average value is 0.75 over a 30-day mission, the RCS fuel requirement for x axis attitude control would be approximately 150 lbs.

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*Jahnke and Emde, op. cit.

**A typical variation in Φ_o and $|\sin 2\Phi_o|$ over one year is shown in Figure 12. A derivation for Φ_o as a function of the orbit inclination, nodal regression and season is given in Appendix A of the technical memorandum cited in footnote *, p. 4.

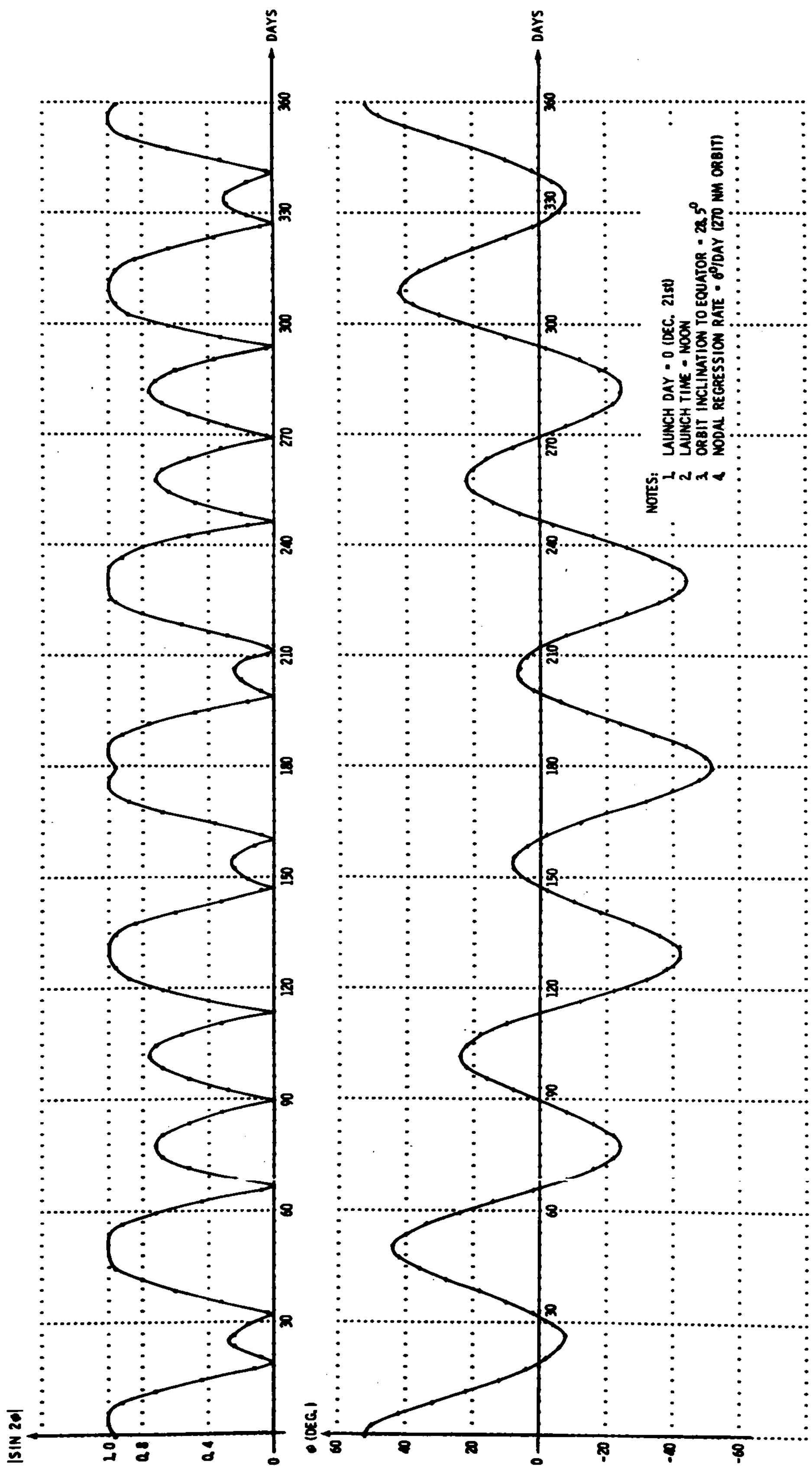

FIGURE 12 - VARIATION IN SOLAR POINTING ANGLE OUT-OF-ORBIT-PLANE DUE TO NODAL REGRESSION AND SEASON

5.0 SUMMARY AND CONCLUSIONS

In this report an approach for stabilizing the AAP 1/2 cluster configuration in a Quasi-Inertial attitude mode has been investigated. The analysis includes an exact solution of the spacecraft equations of motion, an evaluation of typical initialization requirements for the mode and finally an estimate of the RCS fuel requirements including the effect of spacecraft asymmetry.

In the Quasi-Inertial mode the spacecraft principal axis of minimum moment of inertia (x axis) oscillates in the orbital plane about an orientation normal to the sun line. The results of this study indicate that the maximum deviation ($\Delta\Psi_m$) of the x axis from the normal varies between approximately 18° and 60° for the AAP 1/2 cluster configuration depending on the ratio (α) between the peak aerodynamic torque and peak gravity-gradient torque. Due to this deviation fixed solar panels will operate at less than 100% efficiency, specifically $|\cos\Delta\Psi(t)|_{ave.}$ x 100%. A tabulation of $|\cos\Delta\Psi(t)|_{ave.}$ from the curves of $\Delta\Psi(t)$ in Figure 9 over the interval ($T/4 \leq t \leq 3T/4$) is given in Table III. Since it is possible by appropriate initialization to achieve this half orbit on the light side, the results in Table III indicate that the drop in efficiency is only 21% for the maximum value of α considered and this decreases sharply as $\alpha \to 0$.

TABLE III - Summary of Solar Panel Efficiency Factors in Quasi-Inertial Mode

α	$\Delta\Psi_m$	$\lvert\cos\Delta\Psi(t)\rvert_{ave.}$
0	17.8°	0.977
0.03	23.7°	0.993
0.12	38.0°	0.969
0.4	59.1°	0.79

The RCS fuel requirements for the Quasi-Inertial mode are associated with periodic initialization operations and roll control about the x axis due to spacecraft asymmetry. The results of this study indicate that an RCS fuel requirement of 300 lbs./mo. or less appears possible. This is to be compared with 5800 lbs./mo. for a true inertial attitude hold of the AAP 1/2 cluster configuration.*

In summary, the Quasi-Inertial approach offers the possibility of obtaining nearly maximum efficiency from fixed solar panels with relatively low RCS fuel consumption.

ACKNOWLEDGEMENT

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Attachment
Appendices A and B

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*See Appendix B in reference cited in footnote *, page 4.

APPENDIX A

Spacecraft Attitude - Equations of Motion

The purpose of this Appendix is to obtain the differential equations describing the motion of the spacecraft with the principal axis of minimum moment of inertia in the orbital plane. From Newton's second law for rotational motion the basic equation describing the attitude behavior of the spacecraft is

$$\frac{d}{dt}(I\underline{\omega}) = I\dot{\underline{\omega}} + \underline{\omega} \times I\underline{\omega} = \underline{T}_D + \underline{T}_c \qquad \text{(A-1)}$$

This equation expresses the rate of change of angular momentum in terms of $\underline{\omega}$ the spacecraft angular velocity with respect to an inertial reference coordinate system, I the spacecraft inertia tensor, the disturbance torque $\underline{T}_D$ (gravity-gradient and aerodynamic torque) and a control torque $\underline{T}_c$. As shown in Figure A-1 the spacecraft principal axes (xyz) are related to an inertial coordinate system (XYZ) by the Euler rotation sequence (Ψ,θ,Φ). The x axis is assumed to be the principal axis of minimum moment of inertia and the local vertical position of the spacecraft in orbit is defined by the angle η.

In view of Figure A-1 it follows that

$$\underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\Phi & \cos\theta\sin\Phi \\ 0 & -\sin\Phi & \cos\theta\cos\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} \qquad \text{(A-2)}$$

Since the principal axis inertia tensor is given by

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad \text{(A-3)}$$

Appendix A

GRAVITY GRADIENT TORQUE

$$\underline{T}_g = -\frac{3\omega_o^2}{2}\left[\underline{i}\,A(I_z-I_y)+\underline{j}\,B(I_z-I_x)+\underline{k}\,C(I_y-I_x)\right]$$

$$\omega_o^2 = Gm_e/R_h^3$$

$$A = \sin 2\Phi\left[\sin^2(\Psi-\eta)-\sin^2\theta\cos^2(\Psi-\eta)\right] + \cos 2\Phi \sin\theta \sin 2(\Psi-\eta)$$

$$B = \sin\Phi\cos\theta\sin 2(\Psi-\eta)+\cos\Phi\sin 2\theta\cos^2(\Psi-\eta)$$

$$C = \cos\Phi\cos\theta\sin 2(\Psi-\eta)-\sin\Phi\sin 2\theta\cos^2(\Psi-\eta)$$

AERODYNAMIC TORQUE

$$\underline{T}_a = -T_{am}\left|\sin\gamma\right|\left[\underline{i}\,(o) + \underline{j}\,D + \underline{k}\,E\right]$$

$$T_{am} = F_{Dm}\, r_{cp}$$

$$D = \sin\Phi\cos(\Psi-\eta) - \cos\Phi\sin\theta\sin(\Psi-\eta)$$

$$E = \cos\Phi\cos(\Psi-\eta) + \sin\Phi\sin\theta\sin(\Psi-\eta)$$

$$\sin\gamma = \sqrt{1-\cos^2\theta\sin^2(\Psi-\eta)}$$

γ = ANGLE OF ATTACK

$F_{Dm} = \frac{1}{2}\rho V_o^2 C_{Dm} A_m$ = PEAK DRAG FORCE

r_{cp} = DISPLACEMENT OF CP FROM CG ALONG x AXIS

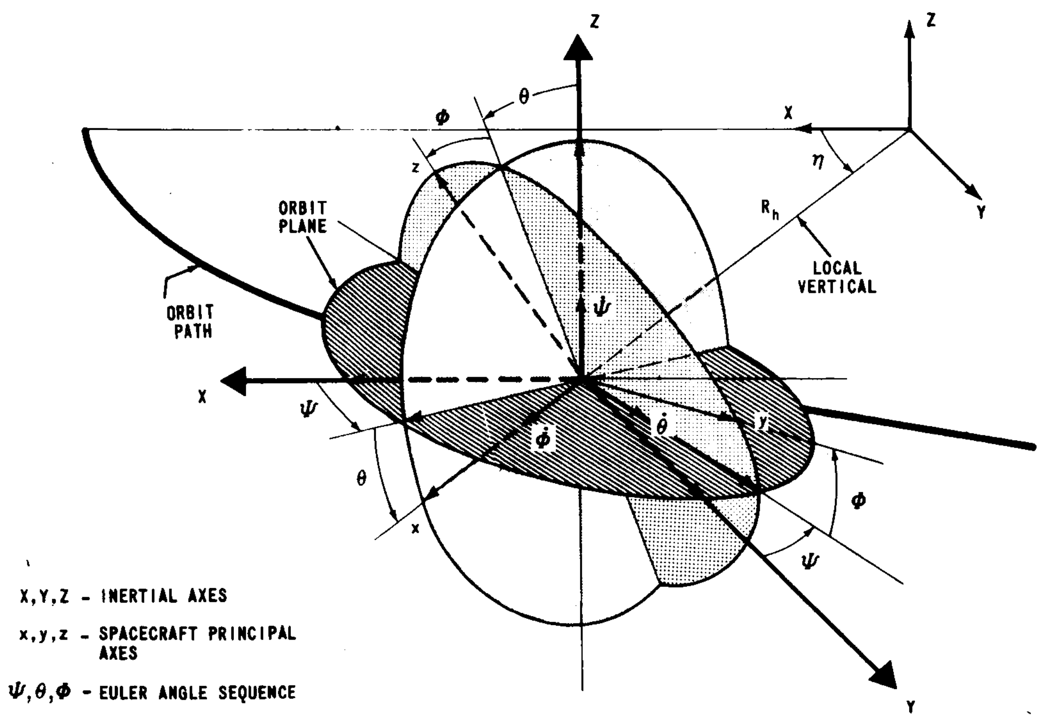

FIGURE A-1 - SPACECRAFT ATTITUDE WITH RESPECT TO INERTIAL REFERENCE; GRAVITY GRADIENT & AERODYNAMIC TORQUE

Appendix A

Equation (A-1) can be written as

$$
I\underline{\dot{\omega}} + \underline{\omega} \times I\underline{\omega} = \begin{bmatrix} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \end{bmatrix} = \begin{bmatrix} T_{gx} + T_{ax} \\ T_{gy} + T_{ay} \\ T_{gz} + T_{az} \end{bmatrix} + \begin{bmatrix} T_{cx} \\ T_{cy} \\ T_{cz} \end{bmatrix} \qquad \text{(A-4)}
$$

The gravity-gradient and aerodynamic torques expressed in spacecraft principal axis coordinates are stated in Figure A-1.*

A general closed form solution of Eq. (A-4) for arbitrary initial conditions is presently unknown. In certain cases however, an analytical solution is possible. One such case involves the free motion ($\underline{T}_c = 0$) of a symmetrical spacecraft ($I_z = I_y$) with the x axis in the orbital plane. The differential equations for this condition can be obtained from Eq. (A-4) after substituting Eq. (A-2) and the expressions for the gravity-gradient and aerodynamic torques given in Figure (A-1). After some manipulation the result is

$$
I_x\left[\ddot{\Phi} - \frac{d}{dt}(\dot{\Psi}\sin\theta)\right] = 0 \qquad \text{(A-5)}
$$

$$
I_z\ddot{\theta} + [I_z\dot{\Psi}\sin\theta + I_x(\dot{\Phi} - \dot{\Psi}\sin\theta)]\dot{\Psi}\cos\theta = -T_{gm}\sin 2\theta\cos^2(\Psi - \eta)
$$

$$
+ T_{am}|\sin\gamma|\sin\theta\sin(\Psi - \eta) \qquad \text{(A-6)}
$$

and

$$
I_z\ddot{\Psi}\cos\theta - [2I_z\dot{\Psi}\sin\theta + I_x(\dot{\Phi} - \dot{\Psi}\sin\theta)]\dot{\theta} = -T_{gm}\cos\theta\sin 2(\Psi - \eta)
$$

$$
- T_{am}|\sin\gamma|\cos(\Psi - \eta) \qquad \text{(A-7)}
$$

---

*The aerodynamic torque given in Figure A-1 is based on the assumption that the center of pressure lies along the x axis and that the airflow along the spacecraft has negligible torque producing effect.

Appendix A

where

$$\sin\gamma = \sqrt{1 - \cos^2\theta \, \sin^2(\Psi-\eta)} \qquad \text{(A-8)}$$

and

$$T_{gm} \equiv \frac{3\omega_o^2}{2} (I_z - I_x) \qquad \text{(A-9)}$$

is the peak gravity-gradient torque and T_{am} is the peak aero-dynamic torque.

It follows from Eqs. (A-5) and (A-6) that the initial conditions $\dot{\theta}_o = \dot{\Phi}_o = 0$ and $\theta_o = 0$ lead to the result that $\ddot{\Phi} = \ddot{\theta} = 0$ for all $t \geq 0$. Consequently a solution for Eqs. (A-5) and Eq. (A-6) is $\theta(t) = 0$ and $\Phi(t)$ = constant. As a result Eq. (A-7) becomes

$$I_z\ddot{\Psi} = -T_{gm} \sin 2(\Psi-\eta) - T_{am}|\cos(\Psi-\eta)|\cos(\Psi-\eta) \qquad \text{(A-10)}$$

Since $\theta(t) = 0$, the spacecraft axis remains in the orbit plane and its motion is described by Eq. (A-10). In using this result for the analysis of the quasi-inertial mode the approximation $(2/\pi)\cos(\Psi-\eta)$ is made for the term $|\cos(\Psi-\eta)|\cos(\Psi-\eta)$.* Then Eq. (A-10) becomes

$$I_z\ddot{\Psi} = -T_{gm} \sin 2(\Psi-\eta) - \alpha T_{gm} \cos(\Psi-\eta) \qquad \text{(A-11)}$$

where $\alpha T_{gm} = 2T_{am}/\pi$ is the equivalent peak aerodynamic torque expressed as a fraction of the peak gravity-gradient torque T_{gm}.

---

*The factor $2/\pi$ corrects for the difference in area under the curves of $\cos(\Psi-\eta)$ and $|\cos(\Psi-\eta)|\cos(\Psi-\eta)$ over any quarter period.

Appendix A

When the spacecraft is not symmetrical ($I_z \neq I_y$), the free solutions, $\theta(t) = 0$ and $\Phi(t) = \Phi_o$ = constant, cannot occur unless $\Phi_o = 0$. This is necessary since the component of gravity-gradient torque T_{gx} acting on the x axis would no longer be zero, as was the case in Eq. (A-5). For the existence of a mode wherein $\Phi(t) = \Phi_o$ = constant and $\theta(t) = 0$, the corresponding angular velocity $\underline{\omega}$ according to Eq. (A-2) must be

$$\underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\Psi} \sin \Phi_o \\ \dot{\Psi} \cos \Phi_o \end{bmatrix} \qquad \text{(A-12)}$$

In view of Eqs. (A-4), (A-12) and the torque expressions in Figure (A-1) the equivalent torque to be counteracted, such that $\omega_x = \dot{\Phi} = 0$,* is given by

$$T_{dx} = (I_z - I_y)\omega_y \omega_z - T_{gx}$$

$$= (T_{gmx}/3)\sin 2\Phi_o[(\dot{\Psi}/\omega_o)^2 + 3\sin^2(\Psi - \eta)] \qquad \text{(A-13)}$$

---

*For spacecraft as large as the Orbital Assembly ($I_z = 2.0 \times 10^6$, $I_x = 0.15 \times 10^6$ slug/ft^2), the maximum angular rate with minimum impulse RCS control in limit cycle operation is adequately low, but not zero. The maximum rate is given by**

$$\dot{\Phi}_m = \left[\int_0^{\Delta t_{min}} T_c dt\right] \Big/ 2I_x$$

For the Apollo CSM, $\int T_c dt \approx 10$ ft.lb.sec., so that $\dot{\Phi}_m \approx 0.002$ deg/sec, which is well below the range of $\dot{\Psi}$ and ω_o to be encountered here.

**See V. E. Haloulakos, "Thrust and Impulse Requirements for Jet Attitude Control Systems", Journal of Spacecraft, Vol. 1, January, 1964, pp. 84-90.

Appendix A

where

$$T_{gmx} = (3\omega_o^2/2)\ (I_z - I_y) \tag{A-14}$$

is the peak gravity-gradient torque about the x axis. The differential equation for Ψ follows from

$$I_y \dot{\omega}_y = I_y \ddot{\Psi} \sin\Phi_o = T_{gy} + T_{ay} \tag{A-15}$$

and

$$I_z \dot{\omega}_z = I_z \ddot{\Psi} \cos\Phi_o = T_{gz} + T_{az} \tag{A-16}$$

which yield

$$\ddot{\Psi} = \dot{\omega}_z \cos\Phi_o + \dot{\omega}_y \sin\Phi_o$$

$$= \left(\frac{T_{gz}}{I_z}\right) \cos\Phi_o + \left(\frac{T_{gy}}{I_y}\right) \sin\Phi_o + \left(\frac{T_{az}}{I_z}\right) \cos\Phi_o + \left(\frac{T_{ay}}{I_y}\right) \sin\Phi_o \tag{A-17}$$

Substitution from Figure (A-1) for the torque expressions yields

$$\ddot{\Psi} = -\left(\frac{T_{gmz}}{I_z} \cos^2\Phi_o + \frac{T_{gmy}}{I_y} \sin^2\Phi_o\right) \sin 2(\Psi - \eta)$$

$$- T_{am}\left(\frac{\cos^2\Phi_o}{I_z} + \frac{\sin^2\Phi_o}{I_y}\right) \left|\cos(\Psi-\eta)\right| \cos(\Psi-\eta) \tag{A-18}$$

where T_{am} is the peak aerodynamic torque and

$$T_{gmz} = (3\omega_o^2/2)\ (I_y - I_x) \tag{A-19}$$

and

$$T_{gmy} = (3\omega_o^2/2)\ (I_z - I_x) \qquad \text{(A-20)}$$

are the peak gravity-gradient torques on the z and y axes respectively. This result can be reduced to a form similar to Eq. (A-11) with the approximation $(2/\pi)\cos(\Psi-\eta)$ for the term $|\cos(\Psi-\eta)|\cos(\Psi-\eta)$.* This yields

$$\ddot{\Psi} = -\ B'\ \sin 2(\Psi-\eta)\ -\ \alpha' B'\ \cos(\Psi-\eta) \qquad \text{(A-21)}$$

where

$$B' = (3\omega_o^2/2)k_z'\ (1 + \Delta) \qquad \text{(A-22)}$$

and

$$\alpha' = \alpha(1 + \Lambda)/(1 + \Delta) \qquad \text{(A-23)}$$

The constants B' and α' are related to Φ_o and the moments of inertia as follows

$$\Delta = (\mu - 1)\sin^2\Phi_o \qquad \text{(A-24)}$$

$$\Lambda = k_{yz}'\ \sin^2\Phi_o \qquad \text{(A-25)}$$

$$\mu = k_y'/k_z' \qquad \text{(A-26)}$$

$$k_y' = (I_z - I_x)/I_y \qquad \text{(A-27)}$$

$$k_z' = (I_y - I_x)/I_z \qquad \text{(A-28)}$$

---

*See footnote on page 4, Appendix A.

Appendix A

and finally

$$k'_{yz} = (I_z - I_y)/I_y \qquad \text{(A-29)}$$

For $I_z = I_y$ it follows that $k'_z = k'_y$, $\mu = 1$, $\Delta = 0$, $k'_{yz} = 0$ and $\Lambda = 0$ so that $\alpha' = \alpha$ and $B' = T_{gmz}/I_z = T_{gm}/I_z$ which is consistent with Eq. (A-11).

When the spacecraft asymmetry is not large the parameters B' and α' will not differ greatly from the symmetrical case ($I_z = I_y$). An indication of the variation in the parameters B' and α' due to spacecraft asymmetry is shown in Table A-1 where the parameters for two asymmetrical configurations, $I_z > I_y >> I_x$ and $I_z > I_y > I_x$, are compared with those for two symmetrical configurations, $I_z = I_y >> I_x$ and $I_z = I_y > I_x$ respectively. In each case $\Phi_o = 45°$ is assumed. The results indicate that the effect of slight spacecraft asymmetry is to decrease α' somewhat whereas B' remains virtually unchanged.

TABLE A-1 - EFFECT OF SPACECRAFT ASYMMETRY ON PARAMETERS B' AND α'

PARAMETERS	(A) $I_z = I_y$	(A) $I_z > I_y >> I_x$	(B) $I_z = I_y$	(B) $I_z > I_y > I_x$
* I_z	2.06×10^6	2.06×10^6	2.06×10^6	2.06×10^6
* I_y	2.06×10^6	1.96×10^6	2.06×10^6	1.56×10^6
* I_x	0.135×10^6	0.135×10^6	0.56×10^6	0.56×10^6
k'_y	0.935	0.982	0.757	0.961
k'_z	0.935	0.886	0.757	0.485
k'_{yz}	0	0.051	0	0.32
μ	1.0	1.110	1.0	1.98
Δ	0	0.055	0	0.49
Λ	0	0.0275	0	0.16
α'/α	1.0	0.974	1.0	0.78
** B'/B_o	0.935	0.934	0.757	0.723

*UNITS: SLUG - FT^2

**$B_o = 3\,\omega_o^2/2$

APPENDIX B

Reduction of Elliptic Integral to Normal Form

The normal form for the elliptic integral of the first kind is

$$F(k_e, \beta) = \int_0^\beta \frac{d\sigma}{\sqrt{1 - k_e^2 \sin^2 \sigma}} \tag{B-1}$$

An equivalent expression for the integral with the radical in polynomial form can be obtained by defining

$$z = \sin\sigma \tag{B-2}$$

where

$$d_z = \cos\sigma \, d\sigma = \sqrt{1 - z^2} \, d\sigma \tag{B-3}$$

Substitution into Eq. (B-1) yields

$$F(k_e, \beta) = \frac{1}{k_e} \int_0^{z_1} \frac{d_z}{\sqrt{R(z)}} \tag{B-4}$$

where

$$R(z) = (1 - z^2)\,(1/k_e^2 - z^2)$$

$$= (z + 1/k_e)(z + 1)(z - 1)(z - 1/k_e) \tag{B-5}$$

and $z_1 = \sin\beta$.

Appendix B

The integral

$$G(\delta) = \int_0^{\delta} \frac{d\psi}{\sqrt{1 - k_a^2 \, (\sin\psi + \alpha/2)^2}} \qquad \text{(B-6)}$$

is also an elliptic integral of the first kind although not in normal form unless $\alpha = 0$. An equivalent expression with the radical in polynomial form follows after defining

$$y = \sin\psi \qquad \text{(B-7)}$$

where

$$dy = \cos\psi \; d\psi = \sqrt{1 - y^2} \; d\psi \qquad \text{(B-8)}$$

Substitution into Eq. (B-6) yields

$$G(\psi) = G(y_1) = \frac{1}{k_a} \int_0^{y_1} \frac{dy}{\sqrt{P(y)}} \qquad \text{(B-9)}$$

where

$$\begin{aligned} P(y) &= (1 - y^2)[1/k_a^2 - (y + \alpha/2)^2] \\ &= \left[y + \left(\frac{1 + k_a \;\alpha/2}{k_a}\right)\right](y+1)(y-1)\left[y - \left(\frac{1 - k_a \;\alpha/2}{k_a}\right)\right] \\ &= (y - a_1)(y - a_2)(y - a_3)(y - a_4) \qquad \text{(B-10)} \end{aligned}$$

The problem here is that the roots of P(y) do not form a symmetrical pattern about the origin, since $a_1 \neq - a_4$.

The elliptic integral in terms of P(y) can be reduced to the normal form in terms of R(z) by using the bilinear transformation.

$$z = D\,\frac{y + A}{y + B} \tag{B-11}$$

In terms of y the transformation is

$$y = \frac{AD - Bz}{z - D} \tag{B-12}$$

where

$$dy = \frac{1}{D}\,\frac{(B - A)}{(1 - z/D)^2}\,dz \tag{B-13}$$

The objective here is to transform the roots of P(y) in order to obtain a symmetrical destribution about the origin as in Eq. (B-5) for R(z). Substitution of Eq. (B-11) into Eq. (B-10) yields

$$P(y) = q^2(z-z_1)(z-z_2)(z-z_3)(z-z_4) = q^2\,R(z) \tag{B-14}$$

where

$$z_1 = D\left(\frac{a_1 + A}{a_1 + B}\right) \tag{B-15}$$

$$z_2 = D\left(\frac{a_2 + A}{a_2 + B}\right) = D\left(\frac{A - 1}{B - 1}\right) \tag{B-16}$$

$$z_3 = D\left(\frac{a_3 + A}{a_3 + B}\right) = D\left(\frac{A + 1}{B + 1}\right) \tag{B-17}$$

$$z_4 = D\left(\frac{a_4 + A}{a_4 + B}\right) \tag{B-18}$$

and

$$q^2 = \frac{(a_1 + B)(a_2 + B)(a_3 + B)(a_4 + B)}{(z - D)^4}$$

$$= \frac{(B - 1)(B + 1)(a_1 + B)(a_4 + B)}{D^4(1 - z/D)^4} \qquad \text{(B-19)}$$

where $a_2 = -1 = -a_3$.

In order to obtain the normal form indicated in Eq. (B-5) it is necessary that $z_2 = -1$, $z_3 = +1$ and $z_1 = -1/k_e = -z_4$. The first two conditions are satisfied by $B = D$ and $AD = 1$.

The third condition requires that

$$1/k_e = z_4 = \frac{D\,a_4 + 1}{a_4 + D} = -\frac{D\,a_1 + 1}{a_1 + D} = -z_1 \qquad \text{(B-20)}$$

or

$$(a_1 + a_4)D^2 + 2(1 + a_1a_4)D + (a_1 + a_4) = 0 \qquad \text{(B-21)}$$

Solution for D yields

$$D = \frac{-2(1 + a_4a_1) + \sqrt{(a_4-1)(1-a_1)(a_4+1)(1+a_1)}}{2(a_1 + a_4)} \qquad \text{(B-22)}$$

An alternate form for this result is obtained by defining

$$n' = \sqrt{-(1 + a_1)(a_4 + 1)} \qquad \text{(B-23)}$$

and

$$n'' = \sqrt{(1 - a_1)(a_4 - 1)} \tag{B-24}$$

so that

$$D = \frac{n''^2 + n'^2 + 2n''n'}{n''^2 - n'^2} = \frac{n'' + n'}{n'' - n'} \equiv \frac{1}{n} \tag{B-25}$$

Thus, the transformation in Eq. (B-11) can be written as

$$z = \frac{Dy + DA}{y + D} = \frac{Dy + 1}{y + D} = \frac{y + n}{1 + ny} \tag{B-26}$$

Substitution of the results in Eqs. (B-13), (B-14), (B-19) and (B-26) into Eq. (B-9) leads to the following relationship for the elliptic integral analogous to the form in Eq. (B-4)

$$\frac{1}{k_a} \int_0^{y_1} \frac{dy}{\sqrt{R(y)}} = \frac{1}{k_a} \left\{ \frac{1}{m} \left[\frac{1}{k_e} \int_0^{z_1} \frac{dz}{\sqrt{R(z)}} \right] \right\} \tag{B-27}$$

where

$$z_1 = \frac{y_1 + n}{1 + ny_1} \tag{B-28}$$

and

$$m = \frac{\sqrt{(B - 1)(B + 1)(a_1 + B)(a_4 + B)}}{D(B - A)k_e}$$

$$= \frac{\sqrt{(1-n^2)(1+na_1)(1+na_4)}}{(1-n^2)k_e} = \sqrt{\frac{(1+na_1)(1+na_4)}{(1-n^2)k_e^2}} \tag{B-29}$$

since $D = 1/n = B$ and $A = 1/D = n$.

Appendix B

In view of Eqs. (B-20) and (B-25) it follows that

$$k_e = \frac{1 + na_4}{a_4 + n} = - \frac{1 + na_1}{a_1 + n} \tag{B-30}$$

so that

$$m = \sqrt{\frac{-(a_1 + n)(a_4 + n)}{(1 - n^2)}} \tag{B-31}$$

An alternate form for k_e and m can be obtained by defining

$$m' = \sqrt{(1 - a_1)(a_4 + 1)} \tag{B-32}$$

$$m'' = \sqrt{-(1 + a_1)(a_4 - 1)} \tag{B-33}$$

Substitution for n from Eqs. (23), (24) and (25) and further manipulation yields

$$k_e = \frac{m' - m''}{m' + m''} \tag{B-34}$$

and

$$m = \frac{m' + m''}{2} \tag{B-35}$$

In summary, the elliptic integral in Eq. (B-6) can be written in normal form (with arbitrary limits) as

$$\int_{\delta_0}^{\delta_1} \frac{d\psi}{\sqrt{1 - k_a^2(\sin\psi + \alpha/2)^2}} = \frac{1}{k_a m} \int_{\beta_0}^{\beta_1} \frac{d\sigma}{\sqrt{1 - k_e^2 \sin^2\sigma}}$$

$$= \frac{1}{k_a m} [F(k_e, \beta_1) - F(k_e, \beta_0)] \tag{B-36}$$

with the variables ψ and σ related by Eqs. (B-2, (B-7) and (B-26) so that*

$$\sin\sigma = \frac{\sin\psi + n}{1 + n \sin\psi} \tag{B-37}$$

The parameters m, k_e and n can be expressed in terms of k_a and α after substituting a_1 and a_4 from Eq. (B-10) into Eqs. (B-23), (B-24), (B-32) and (B-33). The resulting expressions

$$n' = \sqrt{1 - (1 - \alpha/2)^2 k_a^2} / k_a \tag{B-38}$$

$$n'' = \sqrt{1 - (1 + \alpha/2)^2 k_a^2} / k_a \tag{B-39}$$

$$m' = \sqrt{(1 + k_a)^2 - (k_a\ \alpha/2)^2} / k_a \tag{B-40}$$

and

$$m'' = \sqrt{(1 - k_a)^2 - (k_a\ \alpha/2)^2} / k_a \tag{B-41}$$

can then be used to evaluate m, k_e and n given by *

$$m = \frac{m' + m''}{2} \tag{B-42}$$

$$k_e = \frac{m' - m''}{m' + m''} \tag{B-43}$$

and

$$n = \frac{n'' - n'}{n'' + n'} \tag{B-44}$$

---

*The form of these results is identical to that given by J. Houel, Recuil de Formules et de Tables Numeriques, Gauthier-Villars, Paris, 1901, pp. 53, 54.

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